WHAT IS A CALABI-YAU SPACE?

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ABSTRACT

A pedagogical discussion is given of some of the mathematical tools needed for the investigation of complex manifolds with Ricci flat Kähler metrics.

Recent investigations of possible background configurations for superstrings have shown that the spacetime geometry is highly constrained.[1] If one looks for a configuration of the form $M^4 \times K$ where M^4 is four dimensional Minkowski space and K is a compact Riemannian six manifold, then the only known way of satisfying these constraints is for K to be what is called a Calabi-Yau space. This means:

Definition: A Calabi-Yau space is a compact, three dimensional complex manifold with a Ricci flat Kähler metric.

The mathematical machinery needed for the study of Calabi-Yau spaces include complex manifold theory

and algebraic geometry. Since these fields are not yet familiar to many physicists, I will try to give an introduction to the basic concepts involved.[2] I will not go into the details of why these spaces are of interest for superstrings. The mathematics that will be used has obvious generalizations to higher dimensions. However to keep the discussion as concrete as possible, I will restrict myself to six real (or three complex) dimensional manifolds. <u>All manifolds will be assumed to be compact and have only one connected component.</u>

In order to minimize the confusion which might arise from introducing many new definitions, let me begin by mentioning the relation which exists between the concepts we are going to discuss.

Consider the set A of all six dimensional real manifolds. Contained within A lies the subset B of manifolds which admit a complex structure and hence can be viewed as complex three dimensional manifolds. There is a subset $C \subset B$ consisting of those complex manifolds which admit Kähler metrics, and finally a subset $D \subset C$ of manifolds admitting Ricci-flat Kähler metrics. All of these subsets are proper, in the sense that there are manifolds in A but not B, in B but not C, etc. The sets A, B, and C all contain an infinite number of manifolds. However D is believed to be just a finite set. Although the number of manifolds in D is not yet known, a reasonable guess[3] seems to be about 10,000. (Of course the number of phenomenologically interesting Calabi-Yau spaces is considerably smaller.)

Let me begin by reviewing some general properties of real manifolds, and then specialize to complex, Kähler, and finally Ricci flat Kähler spaces.

A. <u>Real Manifolds</u>

Let M be a real six dimensional manifold. We are interested in studying the cohomology of M. This is conveniently described in terms of differential forms. (This is known as de Rham cohomology.) Let x^μ be local coordinates on M and let ω be a p form:

$$\omega = \omega_{\mu\ldots\nu}\, dx^\mu \wedge \ldots \wedge dx^\nu \tag{1}$$

where the coefficients $\omega_{\mu\ldots\nu}$ have p indices. The natural derivative operator on forms is the exterior derivative or curl:

$$d\omega = \frac{\partial\omega_{\mu\ldots\nu}}{\partial x^\sigma}\, dx^\sigma \wedge dx^\mu \wedge \ldots \wedge dx^\nu \tag{2}$$

So d maps a p form into a p+1 form. Since partial derivatives commute, $d^2 = 0$. We now introduce the following definitions:

<u>Definition</u>: A p form ω is <u>closed</u> if $d\omega = 0$. A p form ω is <u>exact</u> if $\omega = d\alpha$ for some globally defined p-1 form α.

Notice that every exact form and every six form on M is automatically closed.

One can show that on $\mathbb{R}^n$ every closed form is exact. Hence, locally on M, one can express any closed form ω as $\omega = d\alpha$. But this is not in general true globally. The obstruction to doing so is the <u>p^{th} cohomology group</u>* $H^p(M)$:

<u>Definition</u>: $H^p(M) \equiv \{$all closed p forms where two forms ω and ω' are considered equivalent if $\omega-\omega'$ is exact$\}$

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*This is cohomology over the field of real numbers. We will soon consider cohomology over the complex numbers. One can also define a more primitive notion of cohomology using only integer coefficients.

One often writes this as a quotient:

$$H^p(M) = \frac{\text{closed } p \text{ forms}}{\text{exact } p \text{ forms}}$$

For each p, $H^p(M)$ is a real vector space. The dimension of this space is called the <u>p^{th} Betti number</u> b_p. H^0 is just the space of constant functions, so $b_0 = 1$. (This would no longer be true if we considered disconnected manifolds.) The <u>Euler number</u> χ of M is defined to be the alternating sum of the Betti numbers:

$$\chi(M) = \sum_{p=0}^{6} (-1)^p b_p \tag{3}$$

Given a Riemannian metric $g_{\mu\nu}$ on M, one obtains an inner product on the space of p forms:

$$\langle\omega|\tau\rangle = \int \omega^{\mu\cdots\nu}\tau_{\mu\cdots\nu}\sqrt{g}\,d^6x \tag{4}$$

where the indices of ω are raised with the metric. Using this inner product, one can define the adjoint $d^\dagger$ of d which maps p forms to p-1 forms. $d^\dagger$ is just the covariant divergence of the form. The <u>Laplacian</u> is defined to be $\Delta_d = dd^\dagger + d^\dagger d$ and solutions to $\Delta_d\omega = 0$ are called <u>harmonic.</u> Since

$$\langle\omega|\Delta_d\omega\rangle = \|d^\dagger\omega\|^2 + \|d\omega\|^2 , \tag{5}$$

one has immediately that a form is harmonic if and only if it is both curl free and divergence free.

A fundamental theorem in this subject is the following

<u>Hodge Theorem (Real Version)</u>

Every p form ω has a unique decomposition:

$$\omega = \alpha + d\beta + d^\dagger\gamma \tag{6}$$

where α is harmonic, β is a p-1 form and γ is a p+1 form.

Notice that the three terms in the above decomposition are orthogonal with respect to the inner product (4) e.g. $\langle d\beta|d^\dagger\gamma\rangle = \langle d^2\beta|\gamma\rangle = 0$. If ω is closed then the last term must vanish (since $dd^\dagger\gamma = 0$ implies $\langle d^\dagger\gamma|d^\dagger\gamma\rangle = 0$). Thus the Hodge theorem implies that <u>there is a unique harmonic representative for each equivalence class in $H^p(M)$.</u> In other words, given any curl free p form ω, one can add to it the curl of some p-1 form so that the sum is divergence free. Notice that if you change the metric then the harmonic forms will change. However the total number of linearly independent harmonic p forms will not change and will always equal b_p.

If the manifold M with metric $g_{\mu\nu}$ is orientable, then there exists a covariantly constant volume form i.e. a 6 form $v_{\mu\cdots\nu}$ normalized so that $v_{\mu\cdots\nu}v^{\mu\cdots\nu} = 6!$ One can use this form to define the dual of a p form

$$*\omega_{\mu\cdots\nu} = \frac{1}{p!}\,\omega^{\rho\cdots\sigma}v_{\rho\cdots\sigma\,\mu\cdots\nu} \tag{7}$$

If ω is harmonic, then the 6-p form $*\omega$ is also harmonic. This implies <u>Poincaré duality:</u> $b_p = b_{6-p}$.

The last concept I wish to review about real manifolds is the <u>holonomy group</u>. Given a connection i.e. a derivative operator on M one defines the holonomy group as follows. Fix a point $p \in M$ and consider any closed curve γ containing p. Take any basis for the tangent space at p and parallel transport it along γ. The result will be a new basis at p which is related to the original one by an element of $GL(6,\mathbb{R})$. Repeating for all closed curves γ, one obtains a subgroup of $GL(6,\mathbb{R})$. This subgroup is independent of the original point p and is called the holonomy group of the connection. If the connection preserves a metric (and the manifold is orientable) then the holonomy group will

be a subgroup of SO(6).

We now pause to consider a few examples. Since the Betti numbers are independent of the metric, one can compute them by picking a simple metric and counting the number of harmonic forms with respect to this metric.

1) $\underline{S^6}$ Consider the standard metric of constant curvature. It is not hard to show that with respect to this metric the only harmonic forms are the constant functions and multiples of the volume form. Thus $b_0 = b_6 = 1$, $b_i = 0$ $1 \le i \le 5$, and the Euler number is $\chi = 2$. For the connection defined by this metric, the holonomy group is SO(6), since there are no subspaces of the tangent space left invariant under parallel transport.

2) $\underline{S^3 \times S^3}$ Consider the standard metric on each S^3. In addition to $b_0 = b_6 = 1$, one now has $b_3 = 2$ since the volume form on each S^3 is harmonic. The remaining Betti numbers vanish and $\chi = 0$. The holonomy group is SO(3) x SO(3).

3) $\underline{S^2 \times S^2 \times S^2}$ This is similar to example 2. Consider the standard metric on each S^2. One now has $b_0 = b_6 = 1$, $b_2 = b_4 = 3$, $b_1 = b_3 = b_5 = 0$. Hence $\chi = 8$. The holonomy group is SO(2) x SO(2) x SO(2).

4) $\underline{T^6}$ The six torus is not as trivial as one might think. With respect to a flat metric, all the constant forms are harmonic. Hence $b_0 = b_6 = 1$, $b_1 = b_5 = 6$, $b_2 = b_4 = 15$, $b_3 = 20$. This implies $\chi = 0$. The holonomy group of a flat connection is of course just the identity element.

These four examples were not picked completely at random. As we will see, they are simple examples of the four classes of manifolds mentioned earlier: S^6 cannot be viewed as a complex three manifold, $S^3 \times S^3$ is a complex manifold but does not admit any Kähler metric, $S^2 \times S^2 \times S^2$ admits Kähler metrics but not one which is Ricci flat, T^6 admits a Ricci flat Kähler metric.

B. <u>Complex Manifolds</u>

A <u>complex structure</u> on a real manifold M is a tensor field $J^\mu{}_\nu$ satisfying $J^\mu{}_\nu J^\nu{}_\sigma = -\delta^\mu{}_\sigma$ and a certain integrability condition. This integrability condition is precisely what is needed so that one can introduce local complex coordinates z^j on M so that the transition functions between different coordinate patches are holomorphic.* A <u>complex manifold</u> is simply a real manifold with a complex structure.

In order for a complex structure to exist M must clearly be even dimensional. It must also be orientable since the form $dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \wedge dz^3 \wedge d\bar{z}^3$ does not change sign under holomorphic change of coordinates and hence defines an orientation on M. But not every even dimensional orientable manifold admits a complex structure. In general it is a difficult mathematical problem to determine whether a given real manifold is also a complex manifold. The fact that S^6 does not admit a complex structure was realized relatively recently.[4] It is now known that the only sphere S^n which admits a complex structure is the two sphere S^2 which is the complex projective plane CP^1. One can show that the product of two odd dimensional spheres $S^p \times S^q$ always admits a complex structure. So $S^3 \times S^3$, $S^2 \times S^2 \times S^2$ and T^6 are all complex manifolds.

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*Given a six dimensional real analytic manifold, one can always let the coordinates become independent complex variables and obtain a six complex dimensional manifold. This is not what is being considered here. Here one obtains a <u>three</u> complex dimensional manifold.

Two complex manifolds M and N are said to be equivalent if there exists a one-to-one, onto map $\varphi: M \to N$ such that when expressed in local complex coordinates, φ and φ^{-1} are holomorphic. It is important to keep in mind that a given real manifold may give rise to inequivalent complex manifolds. In other words, it may admit more than one complex structure. In fact, one can often continuously deform the complex structure. A simple example is given by the torus. Consider the complex plane $\mathbb{C}$ and pick a complex number $z = x+iy$ with $y > 0$. Now take the quotient of $\mathbb{C}$ by the lattice generated by $\vec{e}_1 = (1,0)$, $\vec{e}_2 = (x,y)$. The result is a complex one dimensional manifold T_z. For any z, the underlying real manifold is diffeomorphic to the two torus. However it is easy to show that T_z is equivalent to $T_{z'}$ only if $z' = (az+b)/(cz+d)$ with a,b,c,d integers and ad-bc = 1. More generally, on a two sphere with n handles ($n > 1$) there is a 3n-3 (complex) dimensional space of complex structures.

We now want to repeat our discussion of forms and cohomology for complex manifolds. Consider a three dimensional complex manifold M with local coordinates (z^1,z^2,z^3). The tangent space to M is a six dimensional complex vector space. (This is the complexification of the six dimensional real tangent space of the underlying real manifold.) The cotangent space is spanned by dz^j, $d\bar{z}^{\bar{j}}$ $j,\bar{j} = 1,2,3$. We define a (p,q) form to be a form which is p-fold linear in the dz^j's and q-fold linear in the $d\bar{z}^{\bar{j}}$'s:

$$\omega = \omega_{j\ldots k\bar{j}\ldots\bar{k}}\, dz^j\wedge\ldots\wedge dz^k\wedge d\bar{z}^{\bar{j}}\wedge\ldots\wedge d\bar{z}^{\bar{k}} \tag{8}$$

where $\omega_{j\ldots k\bar{j}\ldots\bar{k}}$ has p unbarred and q barred indices.

There are two analogs of the operator d for complex manifolds. The first maps (p,q) forms into (p+1,q) forms and is defined by:

$$\partial\omega = \frac{\partial\omega_{j\ldots k\bar{j}\ldots\bar{k}}}{\partial z^{\ell}}\, dz^{\ell}\wedge dz^j\wedge\ldots\wedge d\bar{z}^{\bar{k}} \tag{9}$$

The second maps (p,q) forms into (p,q+1) forms and is just the complex conjugate:

$$\bar{\partial}\omega = \frac{\partial\omega_{j\ldots k\bar{j}\ldots\bar{k}}}{\partial \bar{z}^{\bar{\ell}}}\, d\bar{z}^{\bar{\ell}}\wedge dz^j\wedge\ldots\wedge d\bar{z}^{\bar{k}} \tag{10}$$

Clearly, on any three dimensional complex manifold, ∂ annihilates (3,p) forms and $\bar{\partial}$ annihilates (p,3) forms for any p. Since partial derivatives commute we have $\partial^2 = \bar{\partial}^2 = 0$, $\partial\bar{\partial} + \bar{\partial}\partial = 0$. It is easy to verify that

$$d = \frac{1}{2}(\partial + \bar{\partial}) \tag{11}$$

A form ω of type (p,0) is said to be holomorphic if $\bar{\partial}\omega = 0$. In other words, ω is holomorphic if the coefficient functions $\omega_{j\ldots k}$ are all holomorphic functions of the local coordinates.

The complex cohomology groups are defined in a similar manner to the real case:

Definition: The (Dolbeault) cohomology groups are

$$H^{p,q}_{\bar{\partial}}(M) \equiv \frac{\bar{\partial}\text{ closed }(p,q)\text{ forms}}{\bar{\partial}\text{ exact }(p,q)\text{ forms}}$$

One can show that on $\mathbb{C}^n$ these cohomology groups are trivial, so they again measure global properties of the complex manifold. For each (p,q), $H^{(p,q)}_{\bar{\partial}}$ is a complex vector space.

Given a Hermitian metric $g_{j\bar{k}}$ on M one obtains an inner product on (p,q) forms and can define the adjoints of the operators ∂ and $\bar{\partial}$. $\partial^{\dagger}$ takes the covariant divergence on unbarred indices and $\bar{\partial}^{\dagger}$ takes the covariant divergence on barred indices. One can now introduce two more Laplacians

$\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial$ and $\Delta_{\bar\partial} = \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial$. The powerful Hodge theorem extends to the case of complex forms:

Hodge Theorem (Complex Version)

Every (p,q) form ω has a unique orthogonal decomposition

$$\omega = \alpha + \bar\partial\beta + \bar\partial^\dagger\gamma$$

where $\Delta_{\bar\partial}\alpha = 0$, β is a (p,q-1) form, and γ is a (p,q+1) form.

In particular, if $\bar\partial\omega = 0$ then the last term vanishes and we again have a unique representative α for each cohomology class $H^{p,q}_{\bar\partial}(M)$.

C. Kähler Metrics

For a general complex manifold with Hermitian metric there is no relation between the three Laplacians Δ_d, Δ_∂, and $\Delta_{\bar\partial}$. However there exist a special class of metrics for which all Laplacians agree. These are called Kähler metrics. To define these metrics, let $g_{j\bar k}$ be a Hermitian metric and consider the real (1,1) form:

$$J = i\, g_{j\bar k} dz^j \wedge d\bar z^{\bar k} \tag{12}$$

Definition: A Kähler metric is a Hermitian metric with $dJ = 0$.

If $g_{j\bar k}$ is Kähler, then the closed form J is called the Kähler form.

To develop some intuition for what this definition means, it is convenient to reexpress it in terms of real coordinates on M. Given a complex structure $J^\mu{}_\nu$, a Hermitian metric is just a Riemannian metric $g_{\mu\nu}$ satisfying $J^\mu{}_\rho J^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma}$. A Hermitian metric is Kähler if and only if $J^\mu{}_\nu$ is covariantly constant with respect to the connection defined by $g_{\mu\nu}$. Thus, Kähler metrics are the "nicest" class of metrics on a complex manifold in that the Riemannian structure is compatible with the complex structure.

A simple class of examples of Kähler metrics is furnished by one dimensional complex manifolds. Since dJ is a three form, it follows immediately that every Hermitian metric on a one dimensional complex manifold is Kähler. Of course in higher dimensions this is no longer the case. For any Hermitian metric on a three dimensional complex manifold the volume form V is related to the (1,1) form J Eq. (12) by:

$$V = \frac{1}{3!} J\wedge J\wedge J \tag{13}$$

For a Kähler metric J is closed and hence defines an element of $H^2(M)$. This cohomology class must be non-trivial, for if $J = d\alpha$, then $\int_M J\wedge J\wedge J = 0$ by Stokes' theorem. Similarly, $J\wedge J$ defines a non-trivial cohomology class in $H^4(M)$. Thus we learn that in order for a manifold to admit a Kähler metric, the even Betti numbers must satisfy $b_{2p} \geq 1$. This shows immediately that $S^3 \times S^3$ (which has $b_2 = 0$) does not admit any Kähler metric.

As mentioned earlier, one of the most important properties of Kähler metrics is the following:

Theorem: On a manifold with a Kähler metric

$$2\Delta_d = \Delta_\partial = \Delta_{\bar\partial} \tag{14}$$

This means that a form which is curl free and divergence free with respect to barred indices is also curl free and divergence free with respect to unbarred indices. In particular, consider a holomorphic p form $\bar\partial\omega = 0$. Since ω has no barred indices, $\bar\partial^\dagger\omega = 0$ and hence $\Delta_{\bar\partial}\omega = 0$. By the above theorem, $\Delta_d\omega = 0$. Therefore we have the somewhat

surprising result that holomorphic forms are automatically harmonic with respect to any Kähler metric. Conversely, $\Delta_d \omega = 0$ implies $\bar\partial \omega = 0$, so every harmonic (p,0) form is holomorphic.

The (complex) dimension of $H^{p,q}_{\bar\partial}(M)$ is called the Hodge number* $h^{p,q}$. There is a convenient way of summarizing the cohomology of a manifold which admits a Kähler metric in terms of the Hodge diamond:

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array}$$

This diagram has the following properties:

(1) The sum of the Hodge numbers along the n^{th} row is the n^{th} Betti number:

$$b_n = \sum_{p+q=n} h^{p,q} \tag{15}$$

Notice that this is true even though b_p is the real dimension of $H^p(M)$ and $h^{p,q}$ is the complex dimension of $H^{p,q}_{\bar\partial}(M)$. (See property 2.)

* In some recent physics papers these numbers have been denoted $b_{p,q}$. Here we adopt the more standard mathematical notation.

(2) Complex conjugation maps (p,q) forms into (q,p) forms. Since $\Delta_\partial = \Delta_{\bar\partial}$ we know that ω is harmonic if and only if $\bar\omega$ is. Hence $h^{p,q} = h^{q,p}$ i.e. the diagram is symmetric about the verticle line.

(3) The volume form is a (3,3) form. Since the dual of a harmonic form is also harmonic, $h^{p,q} = h^{3-p,3-q}$. This implies that the diagram is symmetric about the center point.

Properties (1) and (2) give another topological restriction (in addition to $b_{2p} \geq 1$) on the existence of a Kähler metric: b_{2n+1} must be even. Properties (2) and (3) show that only 6 of the Hodge numbers are independent. We will see in the next section that the requirement of a Ricci flat Kähler metric reduces this number even further.

D) Ricci Flat Kähler Metrics

The curvature tensor of a Kähler metric takes a simple form. One can show that the only non-vanishing Christoffel symbols are $\Gamma^j_{k\ell}$ and their complex conjugates $\Gamma^{\bar j}_{\bar k \bar \ell}$ where

$$\Gamma^j_{k\ell} = g^{j\bar m} g_{k\bar m,\ell} \tag{16}$$

and the only non-vanishing components of the curvature tensor are

$$R^j{}_{k\ell\bar m} = -\Gamma^j_{k\ell,\bar m} \tag{17}$$

and those related by symmetry and complex conjugation. Because of this, the Riemann tensor has an extra symmetry:

$$R^j{}_{k\ell\bar m} = R^j{}_{\ell k\bar m} \tag{18}$$

It follows from (16) and (17) that the Ricci tensor of a Kähler metric can always be written locally as

$$R_{j\bar{k}} = \frac{-\partial^2(\ln \det g)}{\partial z^j \partial \bar{z}^{\bar{k}}} \tag{19}$$

Therefore the Ricci form:

$$\boxed{R \equiv iR_{j\bar{k}} dz^j \wedge d\bar{z}^{\bar{k}}} \tag{20}$$

is closed $dR = 0$ and defines an element of $H^2(M)$. This cohomology class turns out to be independent of the Kähler metric you start with and is called the first Chern class c_1. (More generally, c_1 can be defined for any complex vector bundle. Here we are considering the tangent bundle of a complex manifold.) It is obvious that if $c_1 \neq 0$ i.e. the Ricci form is not exact, then there cannot exist a Ricci flat metric. The converse is far from obvious. However, Calabi has shown uniqueness[5] and Yau has proved existence[6] of a Ricci flat metric whenever $c_1 = 0$. (Hence the name Calabi-Yau spaces.*) More precisely, they proved:

Theorem: Given a complex manifold with $c_1 = 0$ and any Kähler metric $g_{j\bar{k}}$ with Kähler form J [Eq. (12)], then there exists a unique Ricci flat Kähler metric $\hat{g}_{j\bar{k}}$ whose Kähler form $\hat{J}$ is in the same cohomology class as J.

Notice that for most physical applications one should not think of the Ricci flat metric as unique. One can change this metric either by changing the cohomology class of J or by changing the original complex structure.

A useful property of manifolds with $c_1 = 0$ is the following: $c_1 = 0$ if and only if there exists a non-vanishing holomorphic three form. (For a simple proof, see Ref. [7].) This three form is in fact covariantly constant with respect to a Ricci flat metric. Since $S^2 \times S^2 \times S^2$ has $b_3 = 0$, it has no harmonic three forms and hence no holomorphic three forms. It then follows that it cannot admit a Ricci flat Kähler metric.

If $c_1 = 0$, the Hodge numbers acquire an extra symmetry. By taking the dual of a harmonic $(p,0)$ form using the covariantly constant three form, one obtains a harmonic $(0,3-p)$ form. (Recall that lowering an index with a Hermitian metric changes its type.) Hence:

(4) If $c_1 = 0$, then $h^{p,0} = h^{0,3-p}$

Notice that if one tries to extend this to other harmonic forms, then one finds that the "dual" of a (p,q) form with $q \neq 0$ is symmetric in some of its indices and hence does not even define a form.

Property (4) reduces the six independent Hodge numbers to just four which we can take to be $h^{0,0}$, $h^{1,0}$, $h^{1,1}$, $h^{2,1}$. However $h^{0,0}$ is just the dimension of the space of constant functions and hence $h^{0,0} = 1$. $h^{1,0}$ usually is also trivial for the following reason. On any Riemannian manifold, one can express the Laplacian on p forms in terms of the covariant derivative ∇_μ and curvature of the metric. (This is called the Weitzenbock identity):

$$\Delta_d = -\nabla^2 + \text{curvature terms} \tag{21}$$

For a one form, the curvature term involves only the Ricci tensor. This shows that on a Ricci flat manifold, every harmonic one form must be covariantly constant and hence non-vanishing. On the other hand, a manifold with $\chi \neq 0$ does not admit any nowhere vanishing one forms. Therefore $\chi \neq 0$ implies $b_1 = 0$ which implies $h^{1,0} = 0$. One can also show that $\chi \neq 0$ implies $\pi_1(M)$ is finite. So there exists a compact simply connected covering space.

*A complex manifold with $c_1 = 0$ is sometimes called a Calabi-Yau manifold.

We have arrived at the following simple result: The cohomology of a Calabi-Yau space with $\chi \neq 0$ is characterized by two integers $h^{1,1}$ and $h^{2,1}$. In particular, the Euler number is $\chi = 2(h^{1,1}-h^{2,1})$. It turns out that $h^{2,1}$ gives the (complex) dimension of the space of complex structures that can be placed on M.* In terms of the compactification of the superstring discussed in [1], $h^{2,1}$ and $h^{1,1}$ give the number of families and anti-families of massless fermions.

Finally, we consider the holonomy group of a Hermitian metric on a complex manifold. If the metric is not Kähler, then the holonomy group will in general be SO(6). This is because the complex structure is not preserved under parallel transport. If the metric is Kähler, then the complex structure is preserved and the holonomy group is contained in U(3). If the metric is Kähler and Ricci flat, then the holonomy group is even further restricted. This can be seen as follows. The Lie algebra of the holonomy group is given by parallel transport around infinitesimal loops.** The change in a vector is then given by the curvature tensor. For a Kähler metric, the change is $R^j{}_{k\ell\bar{m}}v^\ell v^{\bar{m}}$ which are generators of U(3). If we take the trace of these generators and use (18) we obtain:

$$R^k{}_{k\ell\bar{m}} = R^k{}_{\ell k\bar{m}} = R_{\ell\bar{m}} \tag{22}$$

Thus the holonomy group is contained in SU(3) if and only if the metric is Ricci flat and Kähler.

* In general, the Hodge numbers depend on the complex structure. However for compact manifolds admitting Kähler metrics, these numbers do not change under continuous deformations of the complex structure.

** This must be considered at each point of the manifold.

We conclude with a few more examples:

1) CP^3 The complex projective spaces admit Kähler metrics, and have the smallest possible Hodge numbers consistent with this fact: $h^{p,q} = \delta_{p,q}$. Since $h^{3,0} = 0$, there is no holomorphic three form and hence $c_1 \neq 0$. This shows that CP^3 does not admit a Ricci-flat Kähler metric.

2) Consider the submanifold $Y_{4,5}$ of CP^4 defined by

$$\sum_{i=1}^{5} z_i^5 = 0 \tag{23}$$

where z_i are homogeneous coordinates. Let $y_i = z_i/z_5$ for $i = 1,\cdots,4$. Then the holomorphic three form

$$dy_1 \wedge dy_2 \wedge dy_3 / y_4^{\,4} \tag{24}$$

is easily shown to be non-singular and non-vanishing everywhere on $Y_{4,5}$. Hence $Y_{4,5}$ has $c_1 = 0$ and admits a Ricci flat Kähler metric. One can show that $h^{2,1} = 101$ and $h^{1,1} = 1$, so $\chi = -200$. Since $h^{1,1} = 1$, there is a one dimensional space of cohomology classes of the Kahler form J.* In terms of the unique Ricci-flat metric guaranteed by Calabi and Yau's theorem, this corresponds to an overall constant rescaling of the metric. Since $h^{2,1} = 101$, there is a 101 (complex) dimensional space of complex structures on this manifold. This yields a 203 (real) dimensional space of Ricci flat metrics. It is unfortunate that not one of them is known explicitly.

* $h^{1,1} = 1$ means that there is a one complex dimensional space of harmonic (1,1) forms. However, since the Kähler form is real, one has only a one real dimensional space of possible cohomology classes for J.

The different complex structures on this manifold are known explicitly and can be realized as follows. Consider the general fifth order homogeneous polynomial in $\mathbb{C}^5$:

$$f(z^i) = A_{ijk\ell m} z^i z^j z^k z^\ell z^m \quad . \qquad (25)$$

The equation $f = 0$ defines a smooth submanifold M_A of $\mathbb{C}P^4$ provided $df \neq 0$ whenever $f = 0$. One can show that two submanifolds M_A and $M_{A'}$ are always diffeomorphic but have different complex structures unless the tensors A and A' are related by a GL(5,$\mathbb{C}$) transformation. Since there is a 126 (complex) dimensional space of symmetric tensors A and dim GL(5,$\mathbb{C}$) = 25 we obtain 126-25 = 101 inequivalent complex structures.

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2) For a more detailed discussion of the material covered here see P. Griffiths and J. Harris, Principles of Algebraic Geometry (Wiley-Interscience, 1978) Chapter 0; S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. II (Wiley, 1969) Chapter 9.

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